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NONLINEAR PRICING IN MARKETS WITH INTERDEPENDENT DEMAND*

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This paper provides a mathematical framework for modeling demand and determining optimal price schedules in markets which have demand externalities and can sustain nonlinear pricing. These fundamental economic concepts appear in the marketplace in the form of mutual buyers' benefits and quantity discounts. The theory addressing these aspects is relevant to a wide variety of goods and services. Examples include tariffs for electronic communications services, pricing of franchises, and royalty fees for copyrighted material and patents. This paper builds on several previous results from microeconomics and extends nonlinear pricing to markets with demand externalities. The implications of this price structure are compared to results obtained for flat rates and two part tariffs in a similar context. A case study is described in which the results were applied to planning the startup of a new electronic communications service.

(Nonlinear Pricing; Demand Externalities; Telecommunication Networks)

1. Introduction

Overview

This paper describes mathematical models for analyzing prices in markets which are characterized by two important features: nonlinear pricing and

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demand externalities. Nonlinear pricing, defined simply as pricing in which the price per unit depends on the quantity purchased, appears across a wide variety of goods and services in the form of quantity discounts. Such pricing is feasible for goods and services that have no resale market, either because the supplier can monitor consumption and prevent resale or due to high transaction costs that make resale unprofitable. Occasionally, regulated commodities such as electric power are priced with increasing unit prices (e.g., lifeline rates), but such pricing is not common and will not be considered in this paper. Demand externalities arise in any market in which the benefit that one customer receives from purchasing a good or service depends on the number and perhaps the identities of the other customers who purchase that good. Classic examples of such externalities, often referred to as "network externalities," arise in communication services, since consumers' benefits in communications networks depend on their access to communication partners and increase with the size of the network (assuming no congestion). This phenomenon, however, plays an important role in many other markets where factors such as standardization, fashion or word of mouth advertising increase demand.

The goal of this paper is to provide a framework for determining nonlinear price schedules and analyzing their implications in the presence of demand externalities. The models developed account for heterogeneous markets in which customers differ in their preference and consumption levels, which they select so as to maximize benefit minus cost. For the monopolist supplier case, the models allow the determination of optimal profit maximizing price schedules in various scenarios. For several pricing structures, profitability and consumer welfare implications are discussed and compared. In particular we compare profit maximizing nonlinear tariffs to flat rate and two part tariffs (a fixed subscription fee plus a constant usage price). The interdependence of consumers' demand often leads to a "critical mass" phenomenon, in which a coalition of consumers needs to be formed before any subscriber's benefit justifies the subscription charge. Our model enables the calculation of the minimum and maximum coalition sizes and can be used to evaluate their sensitivity to different price structures.

The results in this paper apply to any market in which demand externalities exist and nonlinear pricing can be sustained. Emerging electronic communications markets, most of which meet these criteria, are of particular interest because many new offerings are seeking to achieve a "critical mass" of users. Some of the terminology for this market will be adopted in presenting our results. An application of the results to a case study of a nationwide communications service for the hearing impaired will be presented in the last section of the paper.

As pointed out earlier, however, the theory presented in this paper applies to other markets as well. Two such examples are franchising and licensing of copyrighted material. In typical franchise offerings, such as fast foods, auto repair, or retail goods, economies of scale in advertising and word of mouth effects create an externality through which each outlet benefits from adding

more outlets to the chain. Such expansion will increase sales (assuming no internal competition) and thus the market value of each existing outlet. In pricing licenses for such outlets, the franchise seller can charge a flat monthly rate, a two-part tariff involving a fixed percentage of dollar sales, or a nonlinear tariff in which the fee is a declining percentage of the dollar sales volume. An important issue faced by the franchise seller is how to set the fees so as to encourage expansion of the chain and yet to allow himself to capture a share of the outlets' appreciation due to the expansion.

An analogous situation exists in the licensing of patented devices or in licensing retailers to distribute trademarked merchandise. The popularity of the licensed item, which strongly influences the ultimate value of the license, depends upon how many licensees engage in the distribution of the material. The analysis presented in this paper applies in the startup stages, when the externality effects are positive.

Related Economics Research

This paper builds upon two research areas of microeconomics which have been quite active over the past few years. Notable contributions in nonlinear pricing are by Spence (1977), Goldman, Leland and Sibley (1977), Roberts (1979), Willig (1978) and in the multiproduct context by Mirman and Sibley (1981). Related work in the context of public goods has recently been published by Brito and Oakland (1980). These analyses examine the case of a monopolist supplier offering a good with a unit price that depends upon the quantity purchased. The implications of such pricing in the case of oligopolistic competition have been studied recently by Oren, Smith and Wilson.

Externality effects resulting from demand interdependencies have also been addressed in several contexts. For flat rate and two part tariffs, the implications with respect to critical mass and optimal pricing have been studied by Artle and Averous (1975), Littlechild (1975), Rohlfs (1974), Squire (1973), and Oren and Smith (1981). In this paper, we extend the results obtained by Oren and Smith for the flat rate and two part tariff to the general nonlinear tariff structure. Using the model described by Oren and Smith, we derive the optimal nonlinear prices for a profit maximizing monopoly.

2. Formulation

Specification and Implementation of Nonlinear Prices

Operationally, nonuniform pricing can be offered in various ways. Figure 1 illustrates three alternative types of schedules, which are found in telecommunication markets, utility industries, and photocopier markets. The first is block declining pricing in which the consumer pays different rates for subsequent blocks of consumption. In the second case, the consumer can choose among several types of equipment each priced with a two part tariff consisting of a fixed charge and a corresponding marginal rate. The consumer has the option of choosing a more expensive type of equipment that provides better efficiency and reduced marginal charges. In the third case the consumer can

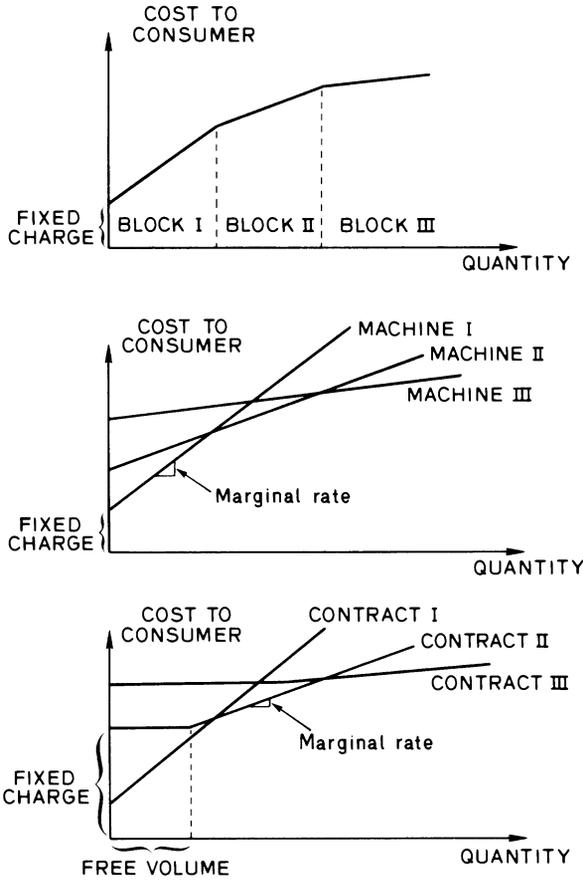


FIGURE 1. Alternative Implementations of Nonuniform Pricing.

choose among alternative contracts consisting of a fixed charge which may include a minimum purchase commitment (free volume) and a corresponding marginal cost above that commitment.

For convenience of analysis, the nonuniform pricing literature typically deals with tariffs that change continuously with quantity. This is equivalent to assuming a continuum of blocks or two part tariffs. This does not limit the usefulness of the results, since piecewise linear price schedules can be used to approximate any general continuous schedule. Faulhaber and Panzar (1977) have investigated the optimal selection of such approximations formulating the problem directly in terms of discrete tariffs, such as those in Figure 1.

Characterizing the Demand

Nonlinear pricing schedules allow the supplier to increase his profits through price discrimination among customers having different demand functions. The fundamental assertion in constructing such pricing schedules is that the supplier knows the distribution of demand functions in the population by

customer type. However, he is unable to discriminate directly among customers according to their type, either because he is prohibited from doing so (by regulation) or he cannot identify individual customers' types. Consequently, he will attempt to discriminate according to amount purchased and will rely on the customers' self-selection of purchase quantity to induce partial discrimination.

Throughout the paper we shall neglect income effects, which is reasonable in markets for which typical consumption levels correspond to a small proportion of consumers' incomes. We shall also ignore the possible negative externalities due to congestion or internal competition and assume that the prices of all alternative modes of satisfying the consumers' demand for service (other than the one under consideration) are fixed.

In the absence of externalities, an individual consumer's demand may be characterized in terms of a function $W(q, t)$, representing his maximum willingness to pay for the first q units of consumption, ($W(0, t) = 0$), or equivalently in terms of $w(q, t) = \partial W(q, t) / \partial q$, which is the marginal willingness to pay for the q th unit. In this formulation, the parameter t is an index (with density function $g(t)$) that identifies customers' types. It is assumed that the density $g(t)$ is known to the supplier, along with the marginal willingness to pay function $w(q, t)$, which is differentiable and satisfies $\partial w / \partial q < 0$, $\partial w / \partial t < 0$.

Without loss of generality, we may assume that t is uniformly distributed on the interval $[0, 1]$. This follows from the monotonicity of the marginal willingness to pay function with respect to the consumer's index, which is assumed throughout the non-uniform pricing literature. If the scaling of the customer index θ is arbitrary, the willingness to pay function $W(\cdot)$ can be equivalently redefined in terms of any strictly monotone transformation of θ . One such transformation is to label each customer with $t = G(\theta)$, which equals the fraction of customers with original indices less than θ . This yields a set of indices t having a uniform distribution over the interval $[0, 1]$. The transformed willingness to pay function $\hat{W}(q, t) = W(q, G^{-1}(t))$ has the desired properties (for convenience we will suppress the $(\hat{\cdot})$).

Demand externalities are included in this model by assuming that the willingness to pay functions also depend upon the identities of the other purchasers (but not upon the quantity they purchase). Let $Y \subseteq [0, 1]$ be the set of t indices corresponding to the customers who have "subscribed" to the service. Then we let $w(q, t, Y)$ and $W(q, t, Y)$ be the corresponding willingness to pay functions for any fixed Y . Again we assume that $\partial w / \partial q < 0$ and $\partial w / \partial t < 0$ hold for each Y and further that

$$W(q, t, Y_1) \geq W(q, t, Y_2) \quad \text{and} \quad w(q, t, Y_1) \geq w(q, t, Y_2) \quad \text{for} \quad Y_1 \supset Y_2.$$

i.e., increasing the number of subscribers increases the marginal and total willingness to pay for each purchase quantity. This monotonicity property implies that the demand externalities are always nonnegative, which ignores

possible congestion effects. Including congestion and positive externalities simultaneously makes the analysis excessively complex. Since these two effects become important at different stages of a network's development, they can be addressed by separate models.

Based on the complete set of willingness to pay functions, the supplier chooses the optimal tariff function $R(q)$, defined as the total charge to a customer consuming q . The supplier assumes that each consumer will select a consumption level that maximizes his consumer surplus, i.e., his benefit minus cost. The conditions this implies are referred to as the self-selection conditions, which lead different customers to choose different consumption levels. To simplify the analysis, we will assume that the tariff function $R(q)$ is continuous and twice differentiable for all $q > 0$, but will allow an upward discontinuity at the origin, reflecting a fixed subscription charge (actually it is sufficient to assume that $R(q)$ is upper semi-continuous as done by Goldman et al.). This jump in $R(q)$ may cause the consumer surplus to be negative at the "optimal" consumption levels of some consumers, in which case they will choose not to subscribe to the service. The above characterization leads to a recursive relationship. On the one hand each consumer's willingness to pay and his consumer surplus depend on the subscriber set Y . On the other hand each consumer's decision to be a member of that set depends on his consumer surplus. In the next section we will define the conditions under which there is an equilibrium subscriber set Y that is consistent with both of these relationships.

3. Customer's Self-Selection Conditions

For any given set $Y \subseteq [0, 1]$ and tariff function $R(q)$, let $q^*(t, Y)$ denote the optimal consumption quantity for customer t . That is,

$$q^*(t, Y) = \arg \max_{q \geq 0} \{ W(q, t, Y) - R(q) \} \quad \text{for } t \in [0, 1]. \quad (3.1)$$

To guarantee that $q^*(t, Y)$ is finite, we will assume that $W(q, t, Y)$ satiates at some finite level of q (which may depend on t and Y). In other words, consumer t would not exceed that level of consumption even at zero marginal charge. From a practical point of view, this satiation assumption is quite reasonable, since in most markets there is an implicit user cost (e.g., time) associated with consumption, in addition to the charge levied by the supplier. This cost and the absence of a resale market will prevent an infinite $q^*(t, Y)$ even at zero marginal charge. For example, people do not spend all their time making local telephone calls although they are free.

Let $Y^*(Y)$ denote the set of consumers who would wish to subscribe, given a set Y of current subscribers. A consumer t belongs to $Y^*(Y)$ if and only if the supremum of his consumer surplus (benefit minus cost), given Y , is nonnegative. Thus, for any $Y \subseteq [0, 1]$ we define the set to set mapping

$$Y^*(Y) = \{ t \mid CS(t, Y) \geq 0 \}, \quad (3.2)$$

where

$$CS(t, Y) = \sup_{q>0} \{ W(q, t, Y) - R(q) \}. \tag{3.3}$$

A network is in equilibrium when the set of subscribers is identical to the set of consumers who wish to subscribe. Thus, an “equilibrium” subscriber set Y is defined as a fixed point of the mapping Y^* satisfying the relation $Y = Y^*(Y)$. The existence of a finite $q^*(t, Y)$ guarantees that the mapping $Y^*(Y)$ is well defined. Clearly, the empty set is always a fixed point of Y^* , and indeed it is the only fixed point if the tariff is so high that all consumers prefer not to subscribe, regardless of Y . In order that a fixed point Y be nonempty, we need to assume that there is some nonempty set of subscribers X such that $Y^*(X) \supseteq X$, i.e., no member of X wants to leave and some nonsubscribers may wish to join. The existence of a nonempty fixed point of Y^* then follows from the monotonicity of $Y^*(Y)$ and is proved formally in Proposition 1 in the Appendix.

Note that if $R(0^+) = 0$, i.e., subscription is free, then $Y^*(Y) = [0, 1]$ for all Y so $[0, 1]$ is an equilibrium subscriber set. In the more general case where $R(0^+) \geq 0$, we show as part of Proposition 1 that any nonempty equilibrium subscriber set must be an interval $[0, y]$. That is, the subscriber set always consists of all consumers with volume greater than a certain size. Since we need to consider only subscriber sets of the form $[0, y]$ we simplify the notation by replacing Y with the marginal subscriber’s index, y , i.e., $w(q, t, y) \equiv w(q, t; [0, y])$, $q^*(t, y) \equiv q^*(t; [0, y])$ and $CS(t, y) = CS(t; [0, y])$. The equilibrium condition $Y^*(Y) = Y$ can now be replaced by the explicit conditions for the last subscriber y in an equilibrium user set,

$$\begin{aligned} CS(y) = W(q^*(y, y), y, y) - R(q^*(y, y)) &= 0 && \text{if } y < 1, \\ &\geq 0 && \text{if } y = 1. \end{aligned} \tag{3.4}$$

We can also replace (3.1) by the first order necessary conditions

$$\begin{aligned} w(q^*, t, y) &= R'(q^*) && \text{if } q^* > 0, \\ &\leq R'(q^*) && \text{if } q^* = 0. \end{aligned} \tag{3.5}$$

Following Goldman, Leland and Sibley (1977), we assume that the marginal price curve $R'(q)$ crosses the marginal willingness to pay curves $w(q, t, y)$ exactly once from below as shown in Figure 2. This guarantees the uniqueness of $q^*(t, y)$ and implies the monotonicity condition

$$\partial q^*(t, y) / \partial t \leq 0, \tag{3.6}$$

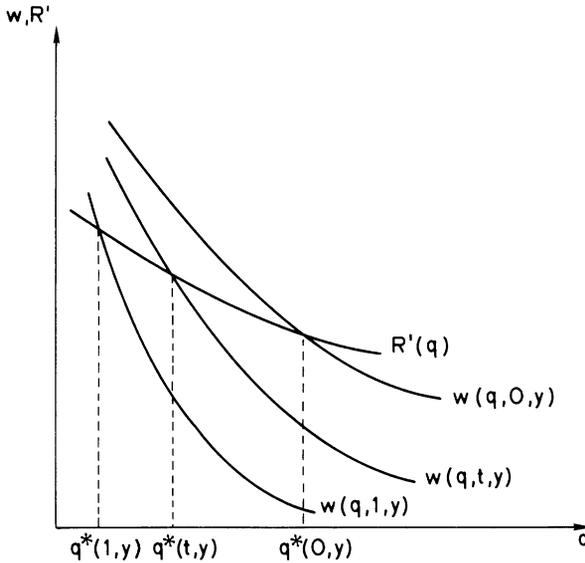


FIGURE 2. Relationship of Marginal Tariff to Demand Curves.

which is equivalent to the second-order necessary condition for (3.1). These results are proved in Proposition 2 in the Appendix.

The monotonicity of $q^*(t, y)$ with respect to t suggests a useful interpretation of the index t . First, we shall define the satiation level $Q(t, y)$, for customers t such that $w(Q(t, y), t, y) = 0$. The level $Q(t, y)$ is simply the $q^*(t, y)$ that would result with no charges. Since $\partial Q(t, y)/\partial t \leq 0$ for any y , the customer index $t \in [0, 1]$ is precisely the customer's fractile ranking with regard to maximum possible consumption $Q(t, y)$. Since $\partial q^*(t, y)/\partial t \leq 0$ for any tariff $R(\cdot)$, this same ranking applies regardless of the tariff. Thus the index t is a direct characterization of each customer's quantity consumption preference in all situations. Consequently, the model parameter t can be determined from empirical data on consumption volumes.

4. Critical Mass and Equilibrium User Sets

While our assumptions so far guarantee a unique quantity selection $q^*(t, y)$ satisfying condition (3.5), further assumptions are needed to determine the values of y that will satisfy condition (3.4). Specifically, for $y \leq 1$, we seek solutions of $CS(y) = 0$. The parameter y captures the network externality effect which manifests itself through the dependence of $W(q, t, y)$ on y . For simplicity, we will assume that for any quantity q , $W(q, t, 0) = W(q, 1, y) = 0$ for all $t, y \in [0, 1]$. That is, the service has no value with zero subscribers, and the lowest volume customer ($t = 1$) derives zero benefit.

It follows that the nonnegative function $W(q, y, y)$ describing the willingness to pay of the last subscriber y , is zero at $y = 0$ and $y = 1$ and will therefore have at least one local maximum in the interval $[0, 1]$.

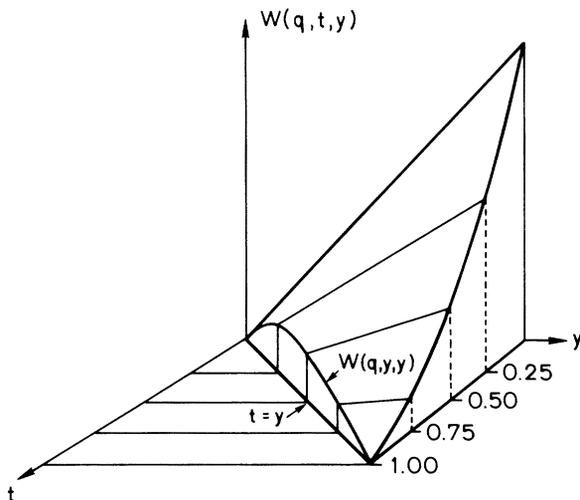


FIGURE 3. The Utility for Consumption Level q , As a Function of Customer Index and Network Size.

Figure 3 illustrates a typical utility function $W(q, t, y)$ and the corresponding $W(q, y, y)$ which is the cross section of $W(q, t, y)$ along the diagonal $t = y$. In this illustration $W(q, y, y)$ is unimodal. However, as shown by Oren and Smith (1981), one can construct examples where $W(q, y, y)$ has multiple local maxima. For analytical convenience we will assume that the values of y at which $W(q, y, y)$ reaches its local extrema are independent of q . (This is true for example if $W(q, t, y)$ is a function of the form $f(q, g(t, y))$ where f is strictly monotone in its second argument for all $q > 0$.) An immediate consequence of the above assumption, proved in Proposition 3 in the Appendix, is that the local extrema of the utility function $W(q, y, y)$ with respect to y coincide with the local extrema of the function $q^*(y, y)$ describing the optimal consumption level of the marginal customer y . This in turn implies that the local extrema $W(q, y, y)$ coincide with those of the last subscriber's consumer surplus function $CS(y)$. Consequently (see Proposition 4 in the Appendix), the roots of the equation $CS(y) = 0$ will occur in pairs of adjacent roots between which the function $CS(y)$ is nonnegative.

Figure 4 illustrates a typical form of $CS(y)$ in relation to $W(q^*(y, y), y, y)$, which illustrates the above remarks. In view of condition (3.4), only values of y in the intervals $[y_1, y_2]$ and $[y_3, y_4]$ define viable subscription levels since only in these intervals does the marginal subscriber have the incentive to subscribe. The points y_1 and y_3 define "Critical Mass" subscription levels. If, for example, the subscription level is below level y_1 , the marginal subscriber has a negative consumer surplus and will therefore maximize his utility by leaving the network. The same thing is true for his predecessor so a chain reaction of subscription cancellations will follow until the network reaches a stable equilibrium (at $y = 0$). On the other hand, if the network reaches the subscription level y_1 , then subsequent customers can obtain a positive con-

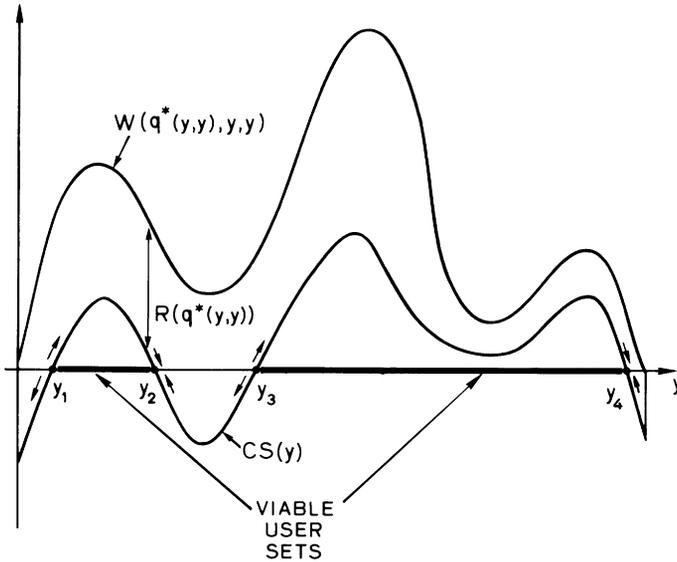


FIGURE 4. Typical Form of the Last Subscriber's Consumer Surplus $CS(y)$.

sumer surplus by subscribing and the network will expand spontaneously to its next "equilibrium user set" $[0, y_2]$. Beyond y_2 , additional customers will have no incentive to subscribe unless the network reaches, by some means, level y_3 from which it will again expand spontaneously to y_4 . Similar phenomena and concepts have been described previously by Artle and Averous (1975), Rohlfs (1974) and by Oren and Smith (1981), in the context of other models and tariff structures.

We also note some additional observations which are evident from Figure 4. First, a fixed charge in the tariff $R(q)$, i.e., a jump at $q = 0$, will always result in a nonzero critical mass level. That is, the subscription level must reach a certain size before the network can be viable. Second, a marginal increase in the equilibrium user set and a reduction in critical mass level result from a reduction in the total charge $R(q^*(y, y))$ paid by the marginal subscriber at that level for his optimal consumption.

5. The Monopoly Profit Maximizing Price Schedule

As shown in the previous section, the consumption level of each subscriber is influenced by the marginal tariff and the network size, while the equilibrium network size is determined by the last subscriber's fee. Both the usage levels and the network size affect the revenues of a monopoly supplier, who controls these quantities by choosing the marginal price schedule $R'(q)$ and the fixed charge $R(0^+)$. In this section, we derive conditions for the optimal schedule $R(q)$ that maximizes the equilibrium net revenues of a monopoly supplier, whose per customer supply cost is given by a function $C(q)$. We assume that

$C(q)$ is increasing and continuously differentiable for $q > 0$, but may have an upward jump $k = C(0^+)$ at the origin.

We will pursue our analysis by first deriving conditions for the optimal marginal tariff $R'(q)$, parametric on a given equilibrium user set y (for notational simplicity, however, we suppress the parameter y in $R(\cdot)$). Then we will obtain conditions for the optimal network size y . To simplify the analysis we will assume from here on that the function $W(q, y, y)$ is unimodal in y . This implies that there can be at most one nonempty equilibrium user set and one corresponding critical mass level.

For any given equilibrium network size y , a monopoly supplier will choose an outlay schedule which maximizes his net revenue $\pi(y)$, where

$$\pi(y) = \int_0^y \{R(q^*(t, y)) - C(q^*(t, y))\} dt, \tag{5.1}$$

while $q^*(t, y)$ satisfies the consumer self-selection conditions (3.5) and (3.6). To determine $R(q)$ uniquely we also need a boundary condition, which is obtained from the fact that the smallest purchaser breaks even. Thus we have

$$R(q_l) = W(q_l, y, y), \tag{5.2}$$

where $q_l = q^*(y, y)$ denotes the smallest subscriber's volume. Similarly $q_u = q^*(0, y)$ will denote the largest subscriber's volume.

As in Goldman et al. (1977), $t^*(q, y)$ is defined such that

$$t^*(q, y) = \max_t \{t \mid q^*(t, y) \geq q\}.$$

For given y , $t^*(q, y)$ corresponds to the subscriber with the smallest potential volume who would still choose to consume at least q units. Using this function, we can change the variable of integration in (5.1), obtaining

$$\pi(y) = \int_{q_l}^{q_u} \{R(q) - C(q)\} (-\partial t^*(q, y) / \partial q) dq. \tag{5.3}$$

Integrating (5.3) by parts yields

$$\pi(y) = [R(q_l) - C(q_l)]y + \int_{q_l}^{q_u} \{R'(q) - c(q)\} t^*(q, y) dq, \tag{5.4}$$

where $c(q) = C'(q)$.

Then, substituting the constraints (3.5) and (5.2) into (5.4) and suppressing

the arguments of $t^*(q, y)$ results in

$$\begin{aligned} \pi(y) = \pi(t^*(\cdot); y) &\stackrel{\Delta}{=} y \{ W(q_l, y, y) - C(q_l) \} \\ &+ \int_{q_l}^{q_u} \{ w(q, t^*, y) - c(q) \} t^* dq. \end{aligned} \tag{5.5}$$

To maximize profits, the supplier first determines the mapping $t^*(q, y)$ which maximizes $\pi(t^*(\cdot); y)$. Then using conditions (3.5) and (5.2) he can obtain the tariff that will induce the subscribers' self-selection corresponding to the optimal mapping $t^*(q, y)$ for the assumed equilibrium network size y . (The optimal y value will be determined later.)

Assuming that the monotonicity constraint (3.6) is inactive, $t^*(q, y)$ can be determined, except for endpoint values, by pointwise maximization of the integrand in (5.5). This yields the condition

$$t^* \partial w(q, t^*, y) / \partial t + w(q, t^*, y) - c(q) = 0, \tag{5.6}$$

which implicitly defines $t^*(q, y)$ or alternatively $q^*(t, y)$. This condition is similar to that obtained by Spence (1977) and by Goldman et al. (1977). In the next paragraph, we will explain how this condition is equivalent to the classical maximum monopoly profit condition, holding simultaneously at each possible consumption level q and for any given market penetration y . We note that the derivative of (5.5) with respect to the boundary point q_l is identically zero. This proves that the interior solution implied by (5.6) is also optimal at the boundary $t = y$, i.e., the optimal q_l satisfies $q_l = \lim_{t \rightarrow y} q^*(t, y)$.

Equation (5.6) can be explained intuitively by interpreting $t^*(q, y)$ as the fraction of potential customers who will purchase at least q units at marginal price $p(q) = R'(q) = w(q, t^*(q, y), y)$ or less. We define an aggregate demand function $N(p, q, y)$, equal to the fraction of customers who will buy at least q units at marginal price p , given market penetration y , which satisfies $N(p(q), q, y) = t^*(q, y)$. Dividing (5.6) through by $p(q) = w(q, t^*, y)$, the equation can be expressed as

$$1 - 1/e_{Np}(q, y) = c(q)/p(q), \tag{5.7}$$

where $e_{Np}(q, y)$ is the price elasticity of $N(p, q, y)$ defined as

$$e_{Np}(q, y) = - \{ p \partial N(p, q, y) / \partial p \} / N(p, q, y). \tag{5.8}$$

Condition (5.7) is exactly the classical profit maximizing monopoly condition, parametric on q and y . In other words, for any market penetration level y , the monopoly profit maximizing price schedule $p(q)$ can be determined by treat-

ing each q th purchase unit as a separate market having a demand function $N(p, q, y)$. Clearly, the largest purchase at any price p and penetration level y is always the value of q for which $N(p, q, y) = 0$. Thus, unless the demand of the largest buyer is inelastic, (5.7) and (5.8) imply that the last unit of the largest purchase size is priced optimally at marginal cost, i.e., $p(q^*(0, y)) = c(q^*(0, y))$. For independent demands, Willig (1978) has shown that the latter property is, in fact, a necessary condition for any price schedule that cannot be Pareto dominated. As shown by Ordover and Panzar (1980), however, this is not true in general when customers' demands are dependent. Yet our results suggest that when demand interdependencies are due to network externalities, Willig's proposition still holds.

As mentioned earlier, our analysis assumes that the monotonicity constraint on $q^*(t, y)$ is inactive. If this constraint becomes active, $q^*(t, y)$ remains constant over some range of t . This case can be treated in a manner analogous to the nonexternality model analyzed by Goldman et al., where it is shown that such a range yields a linear segment in the optimal tariff function $R(q)$. That analysis and the approach used to obtain the linear part of the tariff apply here as well. For further details, we refer the reader to Goldman et al. (1977).

6. Optimal Network Size and Fixed Charge

The optimal equilibrium network size y^* can now be determined from the first order necessary condition $d\pi(y)/dy = 0$ where $t^* = t^*(q, y)$. Using (5.2), (5.6) and (3.5), the above condition can be reduced to the form

$$R(q_t)[1 - e_{wy}] - C(q_t) + \int_0^{q_t} \frac{\partial w(q, t, y)}{\partial y} \Big|_{t=\min\{y, t^*(q, y)\}} dq = 0, \quad (6.1)$$

where e_{wy} is the partial elasticity of the willingness to pay function with respect to customer index for the marginal subscriber, i.e.,

$$e_{wy} = -t[\partial W(q, t, y)/\partial t] / W(q, y, y)|_{q=q_t, t=y}. \quad (6.2)$$

We note that the externality effect, which is reflected by the sensitivity of $w(q, t, y)$ with respect to y , is captured by the integral term in (6.1). Since $\partial w/\partial y \geq 0$ this term will be nonnegative and will have an effect similar to reducing the supplier's provision cost $C(q_t)$ for the marginal subscriber. If there is no externality effect at all, i.e., $\partial w/\partial y = 0$, then the integral term in (6.1) vanishes and (6.1) would reduce to the classical monopoly profit maximization condition for the marginal subscriber's charge $R(q_t)$.

Another important observation that follows from (6.1) concerns the issue of cross subsidy. In the absence of network externality, since $e_{wy} > 0$, it is evident from (6.1) that $R(q_t) \geq C(q_t)$. In other words, the charge paid by the last subscriber is no less than the cost of supply for his optimal consumption

quantity. This is not necessarily true in the presence of positive externalities, as will be demonstrated in the example in the following section. We will construct an example in which the integral term in (6.1) is sufficiently large so as to result in $R(q_t) < C(q_t)$. In fact, the optimal network size y^* and corresponding fixed charge may yield a whole range of t for which $R(q^*(t, y^*)) < C(q^*(t, y^*))$. Customers in this range will obtain the service below cost. The occurrence of this phenomenon depends on the strength of the externality effect. If the effect is sufficiently strong, the potential increase in demand and willingness to pay of the large users, resulting from increased market size, induces the supplier to subsidize the low-volume users, offering them the service below cost. The seller's incentive for doing this is that the fees collected from the increased sales volume of the larger users, due to the subscription of the low-volume users, will more than cover the subsidy cost. The consumers of this extra volume also increase their surplus as a result.

It should be noted that such a cross-subsidization among users may occur only when the supplier's cost function per subscriber contains a fixed cost (usage insensitive). As shown earlier the optimal marginal price is always greater than or equal to the marginal cost. Thus, a monopoly cross-subsidization of low-end users can be implemented only by absorbing part of the fixed supply cost (if there is any). In the context of an electronic mail system, for example, a supplier may find it advantageous to provide basic connection equipment below amortized cost in order to induce greater usage of services on which he will make an overall profit.

For a comparative static analysis of the externality effect we will assume that y^* is the optimal subscription level and let $q_u^* \triangleq q^*(0, y^*)$, $q_l^* \triangleq q^*(y^*, y^*)$. Consider a positive perturbation $\delta(q, t)$ in $\partial w(q, t, y)/\partial y$ around y^* , keeping $w(q, t, y^*)$ unchanged. In view of (6.1), the first order effect on y^* of such a perturbation is given by

$$\left\{ d^2\pi(y^*)/dy \right\}^2 \Delta y^* + \int_0^{q_u^*} t \delta(q, t) \Big|_{t=\min\{y^*, t^*(q, y^*)\}} dq = 0. \tag{6.3}$$

Since y^* is a local maximum of $\pi(y)$, $d^2\pi(y^*)/dy^2 < 0$, hence an increase $\delta(q, t)$ in the sensitivity of $w(q, t, y)$ around y^* will induce an increase ($\Delta y^* \geq 0$) in the optimal network size. Since $w(q, t, y^*)$ did not change, neither will q_l^* , $W(q_l^*, y^*, y^*)$ or the marginal tariff $R'(q)$ corresponding to y^* . But, in view of the results in Section 4, the increase in y^* requires a reduction in $R(q^*)$, which according to the above, can be accomplished only through a reduction in fixed cost $R(0^+)$. We therefore conclude that the network externality induces a profit maximizing monopoly to reduce its fixed charge in order to achieve a larger network size.

7. A Special Case

To illustrate the preceding results more concretely, we consider now the communications system analyzed by Oren and Smith (1981). The marginal

willingness to pay function is of the form

$$\begin{aligned}
 w(q, t, y) &= 2w_0[1 - q/Q(t, y)], & q \leq Q(t, y), \\
 &= 0, & q > Q(t, y),
 \end{aligned}
 \tag{7.1}$$

where

$$Q(t, y) = (2T)y(2 - y)(1 - t).
 \tag{7.2}$$

We will also assume that the cost function is of the form

$$C(q) = k + cq.
 \tag{7.3}$$

The quantity $Q(t, y)$, referred to as the realizable potential volume, is the maximum consumption for customer type t given the network size y . The function $w(q, t, y)$ assumes that the marginal willingness to pay decreases linearly with quantity consumed, where w_0 is the average willingness to pay per unit for all customer types and network sizes.

The realizable potential function $Q(t, y)$ results from integrating a simple density function for communication volume between customer t and each customer ξ , given by $4T(1 - t)(1 - \xi)$. The marginal utility function $w(q, t, y)$ clearly satisfies the monotonicity conditions assumed in our general analysis. Furthermore

$$\begin{aligned}
 W(q, y, y) &= w_0q(2 - q/Q(y, y)), & q \leq Q(y, y), \\
 &= 0, & q > Q(y, y).
 \end{aligned}
 \tag{7.4}$$

Since $Q(0, 0) = Q(1, 1) = 0$, the function $W(q, y, y)$ also vanishes at $y = 0$ and $y = 1$. Finally, since $W(q, y, y)$ is monotone in $Q(y, y)$, it has a maximum in $[0, 1]$, which coincides with the unique local maximum of $Q(y, y)$, independently of q .

Using condition (5.6) for this case, we obtain

$$q^*(t, y) = (1 - t)\gamma Q(t, y) = 2T\gamma(2 - y)y(1 - t)^2,
 \tag{7.5}$$

where

$$\gamma = (1 - c/2w_0).
 \tag{7.6}$$

This leads to the optimal marginal tariff

$$R'(q) = 2w_0[1 - \{q\gamma(1-y)/Q(y, y)\}^{1/2}]. \quad (7.7)$$

Integrating (7.7) with the integration constant determined by (5.2) yields the tariff

$$R(q) = (w_0/3)\{6q - 4[(1-y)\gamma/Q(y, y)]^{1/2}q^{3/2} + \gamma^2(1-y)^2Q(y, y)\}. \quad (7.8)$$

Because of the demand interdependencies, the market penetration also influences the optimal marginal prices. Using (7.4), (7.5) and (7.8) we obtain the distribution of revenues and consumer surplus by subscriber type:

$$R(t) = (4w_0T/3)\gamma y(2-y)\{3(1-t)^2 - 2\gamma(1-t)^3 + \frac{1}{2}\gamma(1-y)^3\} \quad (7.9)$$

and

$$CS(t) = (2w_0T/3)\gamma^2 y(2-y)[(1-t)^3 - (1-y)^3]. \quad (7.10)$$

Integrating (7.2), (7.5), (7.9) and (7.10) over the interval $[0, y]$ of all subscribers gives:

| | |
|-----------------------------|--|
| Realizable potential volume | $T(2-y)^2y^2$, |
| Serviced volume | $\frac{2}{3}T\gamma y(2-y)[1 - (1-y)^3]$, |
| Consumer surplus | $\frac{1}{6}w_0T\gamma^2 y(2-y)[1 - (1+3y)(1-y)^3]$, |
| Revenue | $\frac{2}{3}w_0T\gamma(2-\gamma)y(2-y)[1 - (1-y)^2]$. |

All that remains is to determine the optimal market penetration level y^* . By (5.2), (7.4) and (7.5), we have

$$R(q^*(y, y)) = W(q^*(y, y), y, y) = w_0\gamma Q(y, y)(1-y)[2 - \gamma(1-y)], \quad (7.11)$$

$$e_{wy} = \gamma y / [2 - (1-y)\gamma] \quad (7.12)$$

and

$$\partial w(q, t, y) / \partial y = 4w_0(q / Q(t, y))(1 - y) / y(2 - y). \tag{7.13}$$

Substituting (7.1) through (7.13) into (6.1) and changing the variable of integration from q to $z = q / Q(t, y)$ yields

$$\{ w_0(1 - y) / y(2 - y) \} \left\{ \gamma^2 y Q(y, y) [2y^2 - 5y + 4] + 4 \int_{\gamma(1-y)}^{\gamma} t^*(z) z (dq^* / dz) dz \right\} - k = 0. \tag{7.14}$$

From (7.5) we have $t^*(z) = 1 - z / \gamma$ and $dq^* / dz = 2zQ(y, y) / \gamma(1 - y)$. Substituting these in (7.14) and performing the integration results in

$$(1 - y)y(12 - 15y + 5y^2) / 6 = k / 4\gamma^2 T w_0. \tag{7.15}$$

The optimal equilibrium level y^* can then be determined by intersecting the polynomial on the left of (7.15) with the horizontal line at $k / 4\gamma^2 T w_0$. This will yield two solutions; however, only the larger one corresponds to a maximum. This graphical solution for the optimal network size is illustrated in Figure 5.

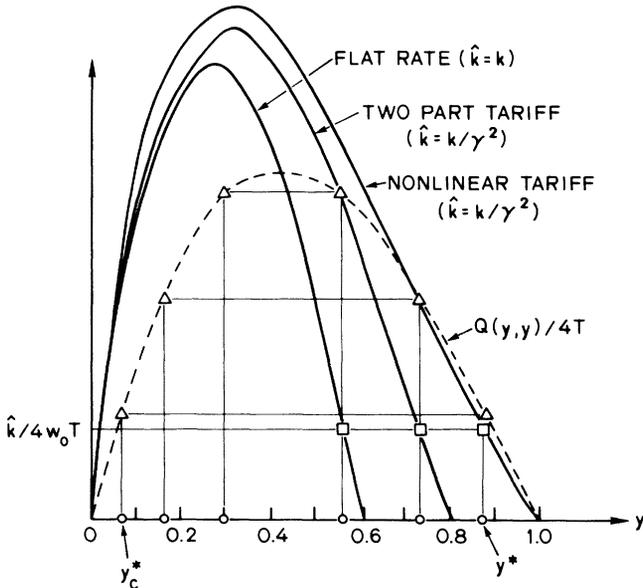


FIGURE 5. Optimal Equilibrium User Set and Critical Mass in Flat Rate, Two Part and Nonlinear Tariffs (Results for Flat-Rate and Two-Part Tariff from Oren and Smith (1981)).

For reference, we also include the corresponding results from Oren and Smith (1981) for the same special case when the tariffs used are the optimal flat rate and two part tariffs. We note in this special case, for the same values of parameters, that the optimal network size corresponding to the nonlinear tariff is larger than that corresponding to the two part or flat rate tariffs. We expect that this would also be true in more general cases. The critical mass level y_c corresponding to y^* is obtained from the breakeven condition

$$R^*(q^*(y_c, y_c)) = W(q^*(y_c, y_c), y_c, y_c), \quad (7.16)$$

where $R^*(q)$ is given by (7.8) with $y = y^*$. The self-selection condition

$$R^{*'}(q^*(y_c, y_c)) = w(q^*(y_c, y_c), y_c, y_c) \quad (7.17)$$

yields

$$q^*(y_c, y_c) = \gamma(1 - y^*) Q^2(y_c, y_c) / Q(y^*, y^*). \quad (7.18)$$

Substituting this into (7.16) yields the relation

$$Q(y_c, y_c) = Q(y^*, y^*), \quad (7.19)$$

from which y_c can be determined. Equation (7.19) implies in this case that the realizable potential volume of the critical mass subscriber, is the same as that of the last subscriber.

The critical level y_c can be obtained graphically, as illustrated in Figure 5. We note that price structures that lead to a larger equilibrium network size also induce a lower critical mass so that a nonlinear tariff structure leads to a critical mass level that is lower than a profit maximizing two part tariff or a flat rate. Although we have substantiated the above assertion only in this special case, we expect that it holds in more general situations. Equation (7.19) is indeed a result of the special assumptions of this example. This result, however, captures the general notion that in presence of network externalities, the demand characteristics of a large user in a small network are similar to those of a small user in a large network. When such symmetry exists in a market we would expect that a larger equilibrium is associated with a lower critical mass level. In this special case it is evident from (7.15) and Figure 4 that when there are no fixed costs of supply ($k = 0$) it is optimal for the supplier to set the fixed charge to zero. This results in an equilibrium network covering the total market and reduces the critical mass to zero.

When $k > 0$, a situation arises in which it is optimal for the supplier to subsidize the low volume users. For demonstration purposes, we will assume that $w_0 T = 5k$ or equivalently that $k = k_0 = w_0 T / 5$, i.e., the average willing-

ness to pay of the average customer for his entire volume in a maximal network is five times the fixed cost of supply. We also assume no marginal supply cost, i.e., $c = 0$, which yields $\gamma = 1$. These parameter values were also used in Oren and Smith (1981), thus enabling us to compare the present results with those obtained for the optimal flat rate and two part tariffs. From (7.15) we then obtain the equation

$$(1 - y)y(12 - 12y + 5y^2) = 0.3,$$

whose larger root is $y^* = 0.875$ i.e., the equilibrium network size cover 87.5% of the potential subscribers. The corresponding critical mass obtained from (7.19) is $y_c = 0.065$.

Substituting the above value of y^* and the assumed ratio $w_0T/k = 5$ in (7.5) (7.9) and (7.10) yields the volume and surplus distributions illustrated in Figures 6 and 7. For comparison we also display the corresponding distributions obtained in Oren and Smith (1981) for the optimal flat rate and two part tariff under the same assumptions. The interesting feature in the surplus distribution for the nonlinear tariff is the fact that an entire segment of subscribers at the low end of the market are subsidized by the supplier. This is optimal since the increased revenues from the other subscribers who communicate with those subsidized, more than cover the subsidy and result in increased profit. This does not occur in this example for two part or flat rate tariffs. In general, however, such a subsidy is possible with two part tariffs.

We summarize in Table 1, for the above example, the various comparisons of the nonlinear tariff with the corresponding results for the flat rate and two part tariffs. Volumes are expressed as multiples of T while monetary values

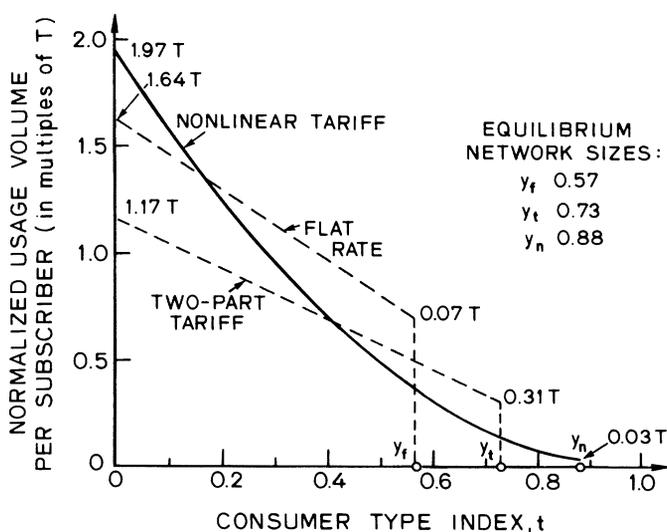


FIGURE 6. Usage Distribution by Type of Subscriber for Three Tariff Structures.

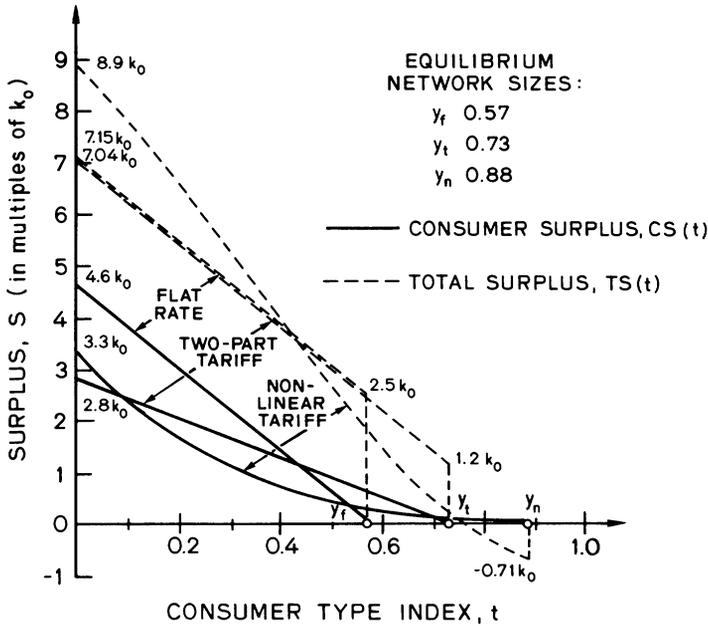


FIGURE 7. Total Surplus and Consumer Surplus by Type of Subscriber for Three Tariff Structures.

TABLE I
Comparison of Three-Tariff Structures

| | <u>Flat Rate</u> | <u>Two Part</u> | <u>Nonlinear</u> |
|-----------------------------|------------------|-----------------|------------------|
| Equilibrium network size | 0.57 | 0.73 | 0.875 |
| Critical mass | 0.29 | 0.16 | 0.065 |
| Subscription fee | $3.52 k_0$ | $1.00 k_0$ | $0.001 k_0$ |
| Realizable potential volume | $0.66 T$ | $0.86 T$ | $0.97 T$ |
| Volume serviced | $0.66 T$ | $0.54 T$ | $0.65 T$ |
| Consumer surplus | $1.32 k_0$ | $1.00 k_0$ | $0.81 k_0$ |
| Net revenue | $1.44 k_0$ | $2.00 k_0$ | $2.34 k_0$ |
| Total Surplus | $2.76 k_0$ | $3.00 k_0$ | $3.15 k_0$ |

are given in multiples of the cost k_0 . While the numerical values in this table have little meaning beyond this example, the direction of change is informative and demonstrates what one would intuitively expect in more general cases. Specifically, the nonlinear tariff reduces total consumer surplus, but is more efficient (in a social welfare sense) and yields a lower subscription fee, a larger network size and a smaller critical mass than either the flat rate of two part tariffs.

8. Case Study Application

The results derived for the special case above have been used in a case study sponsored by the National Telecommunication and Information Administration (NTIA) through SRI International (see Allen et al. (1981)). The

purpose of this study was to examine the commercial feasibility of a nationwide communication system for the hearing impaired at a cost comparable to telephone rates. The system under consideration is based on regional nodes interconnected nationwide through a Value Added Network (VAN), such as GTE Telenet, for example. Each of these nodes contains computer facilities that multiplex the subscribers' terminals, providing them timeshared access to the VAN. A small scale network (DEAFNET) based on this principle was already built by SRI International, establishing its technological feasibility. The goal of the economic analysis was to obtain rough estimates of profitability, social benefit, critical mass and typical charges under various tariff structures. For that purpose we used the simple demand structure implied by (7.1) and (7.2), which requires only a few parameters. The analysis was based on the following inputs.

- The potential market (maximal network size) is two million subscribers, or approximately 1 percent of the U.S. population. The deaf will comprise most of this market, although many hearing persons (relatives, friends) and concerned institutions are expected to subscribe as well.
- Willingness to pay for the services, (w_0), averaged over all potential subscribers (assuming maximum usage and maximal network coverage), is assumed to be 20 cents per call, in addition to any charges for the terminal and for the basic telephone connection.
- Average potential usage in a maximal network, (T), is 200 calls per month per subscriber, assuming free usage (a flat rate tariff). This is close to the current calling rate (local plus long distance) of the average telephone subscriber in the U.S. today.
- Provision cost per subscriber (k_0) is assumed to be \$8.00 per month. This is an optimistic estimate based on projected low cost of computer technology. This cost was assumed to be insensitive to usage which is realistic once the network is set up.

The parameters, w_0 and k_0 , given above are subjective base case estimates, agreed upon by the SRI team involved in this project. In the course of analysis these parameters were varied and the model provided an effective means of evaluating the implications of different parameter assumptions.

Table 2 below shows the base case results obtained for the above input parameter values. These results are based on Table 1, but expanded to present a fuller picture. The modeling assumptions leading to these results, which were presented in this paper, are the following.

- The service will be offered by a monopolist supplier, who will price so as to maximize profits.
- The supplier uses accurate information about preferences of different consumer types and their distribution in the general population in setting prices.
- Any resale market for the offered services is precluded, and the supplier can monitor the usage level of each subscriber. Thus, a nonlinear tariff offering volume discounts is realistic.

The figures in Table 2 should be viewed as a preliminary measure of economic feasibility rather than as a financial analysis in a business sense. A

TABLE 2
Economic Analysis of a Telecommunication Network for the Hearing Impaired

| | Alternative Tariff Structures | | |
|---|-------------------------------|----------------|----------------|
| | Flat Rate | Two Part | Nonlinear |
| Number of Subscribers | | | |
| Potential Market | 2.00M | 2.00M | 2.00M |
| Equilibrium Network Size | 1.14M | 1.46M | 1.76M |
| Critical Mass | 0.58M | 0.32M | 0.13M |
| Subscriber Payments and Revenues | | | |
| Monthly Subscription Fee | \$28.16 | \$ 8.00 | \$ 0.08 |
| Average Monthly Usage Payment | <u>-0-</u> | <u>\$21.92</u> | <u>\$29.19</u> |
| Average Monthly Subscriber Payment | \$28.16 | \$29.92 | \$29.27 |
| Usage | | | |
| Average Monthly Calls per Subscriber | 232 | 148 | 150 |
| Total Annual Call Volume | 3.17B | 2.59B | 3.17B |
| Average Usage Payment per Call | -0- | 14.82 ¢ | 19.46 ¢ |
| Supplier Costs and Revenues | | | |
| Average Monthly Service Cost | \$ 8.00 | \$ 8.00 | \$ 8.00 |
| Total Annual System Costs | \$ 109M | \$ 140M | \$ 169M |
| Total Annual Supplier's Revenues | <u>\$ 385M</u> | <u>\$ 524M</u> | <u>\$ 618M</u> |
| Total Annual Net Revenue | 276M | 384M | 449M |
| Total Annual Consumer Surplus | <u>\$ 253M</u> | <u>\$ 192M</u> | <u>\$ 157M</u> |
| Total Annual (Social) Surplus | \$ 529M | \$ 576M | \$ 606M |

Notes: M = Million; B = Billion.

complete analysis of profit and loss potential would include tax considerations, startup financing costs, and many other factors.

9. Conclusions

In this paper we have presented techniques for optimization and economic analysis of nonlinear tariffs for markets in which consumers' demand functions are interdependent. Specifically, we deal with markets having positive externalities, in which each buyer benefits from an increase in the number of buyers. This type of externality is characteristic of telecommunication networks, where service must be shared by at least two consumers, but also prevails in many other markets for goods and services. A key phenomena that arises in such markets is the existence of a "critical mass" level of subscribers, i.e., a lower bound on the number of buyers that can sustain a given price. Below that level it is not worthwhile for consumers to join the "buyers community" and any previous subscribers prefer to cancel. Once critical mass is exceeded, however, the buyers community will expand to an "equilibrium

user set,” which is a level such that none want to leave and no more want to join.

In our analysis we derived the optimal nonlinear tariff, i.e., volume discount schedule, which induces the consumer behavior that maximizes a monopoly supplier’s net revenue. In comparing that tariff to alternative tariff structures, we observed that the nonlinear tariff is less equitable than the two part tariff with respect to distribution of surplus among the consumers. The nonlinear tariff is more beneficial to the large users, as it gives them a volume discount. This, in turn, motivates them to use the system for a bigger share of realizable potential volume. In fact, the optimal nonlinear tariff is such that the largest user gets his last unit of consumption at cost, so if $c = 0$ he would send his entire volume. The externality effect on the other hand provides an incentive to the supplier to increase the market penetration by lowering the fixed-charge portion of the tariff, making the service affordable to a larger portion of the population.

In terms of achieving critical mass in starting up a new network, the optimal nonlinear tariff provides an improvement over the two-part tariff. This is because the optimal fixed charge level is lower and revenues are obtained to greater extent from marginal charges, thus allowing subscribers to participate more on a “pay as you use” basis. Because the nonlinear tariff can more closely approximate the buyers’ demand functions, higher total surplus and higher supplier profits result. The total consumer surplus, however, is lower than with the two part tariff and the decline is borne by the middle range users who, in a sense, subsidize the entry of the low volume users.

Appendix

PROPOSITION 1. *Let $R(q)$ be a nondecreasing continuous function of q for $q > 0$ with $R(0) = 0$ and $R(0^+) \geq 0$. Also assume that for every t and y there exists a finite optimal consumption level $q^*(t, y)$ defined by (3.1). If there exists a nonempty set X such that $Y^*(X) \supseteq X$, then there exists a nonempty equilibrium subscriber set satisfying (3.1), (3.2) and (3.3). Furthermore, that set is an interval of the form $[0, y]$.*

PROOF. Since $q^*(t, y)$ is finite, the mapping $Y^*(Y)$ is well defined. The existence of a fixed point for that mapping is implied by Tarski’s fixed point theorem for monotone functions on complete lattices (see Birkhoff (1948)). To invoke that result we need to show that $Y^*(Y)$ is monotone, i.e., $Y^*(Y_2) \supseteq Y^*(Y_1)$ for $Y_2 \supseteq Y_1$. For any $t \in Y^*(Y_1)$, we have $W(q, t, Y_2) - R(q) \geq W(q, t, Y_1) - R(q)$ for each q , therefore,

$$\begin{aligned}
 CS(t, Y_2) &= \sup_{q > 0} \{ W(q, t, Y_2) - R(q) \} \\
 &\geq \sup_{q > 0} \{ W(q, t, Y_1) - R(q) \} = CS(t, Y_1) \geq 0.
 \end{aligned}
 \tag{A.1}$$

Thus $t \in Y^*(Y_2)$, proving the requisite monotonicity property.

This property guarantees the existence of a nonempty fixed point since $Y^*([0, 1]) \subseteq [0, 1]$ and by assumption there exists a nonempty X such that $Y^*(X) \supseteq X$.

Next we prove that the equilibrium subscriber set is an interval $[0, y]$. We have assumed that $W(0, t, Y) \equiv 0$ and $\partial w / \partial t = \partial^2 W / \partial q \partial t \leq 0$. These properties imply (using the mean value theorem twice) that $W(q, t_1, Y) \geq W(q, t_2, Y)$ for $t_1 < t_2$. Hence, for $t_1 < t_2$

$$\begin{aligned} CS(t_1, Y) &= W(q^*(t_1, Y), t_1, Y) - R(q^*(t_1, Y)) \\ &\geq W(q^*(t_2, Y), t_1, Y) - R(q^*(t_2, Y)) \\ &\geq W(q^*(t_2, Y), t_2, Y) - R(q^*(t_2, Y)) = CS(t_2, Y). \end{aligned} \tag{A.2}$$

It follows that if $t_2 \in Y^*(Y)$, i.e., $CS(t_2) \geq 0$ then $CS(t_1, Y) > CS(t_2, Y) > 0$ implying $t_1 \in Y^*(Y)$ for all $t_1 \leq t_2$. Consequently $Y^*(Y)$ is an interval $[0, y]$ and any equilibrium subscriber set Y satisfying $Y^*(Y) = Y$ is of the same form. Q.E.D.

PROPOSITION 2. *Let $q^*(t, y)$ satisfy equation (3.4) and assume that $R'(q)$ intersects $w(q, t, y)$ exactly once from below for all $t, y, \epsilon [0, 1]$. Then $q^*(t, y)$ is monotonically decreasing in t . Furthermore, $q^*(t, y)$ satisfies second order necessary conditions for (3.1), namely,*

$$\partial w(q, t, y) / \partial q - R''(q) \leq 0. \tag{A.3}$$

PROOF. Let $q_1^* \triangleq q^*(t_1, y)$. Then from (3.5) and the monotonicity of $w(q, t, y)$ in t it follows that for any $t_2 \geq t_1$,

$$w(q_1^*, t_2, y) - R'(q_1^*) \leq w(q_1^*, t_2, y) - w(q_1^*, t_1, y) \leq 0. \tag{A.4}$$

However, since $R'(q)$ intersects $w(q, t_2, y)$ only once from below at $q^*(t_2, y)$, this implies that

$$q^*(t_2, y) \leq q^*(t_1, y) \quad \text{for } t_2 \geq t_1, \tag{A.5}$$

or equivalently

$$\partial q^*(t, y) / \partial t \leq 0. \tag{A.6}$$

For $q^*(t, y) > 0$ the total differential of the first order condition (3.4) with respect to t yields

$$\frac{\partial q}{\partial t} \left(\frac{\partial w}{\partial q} - R'' \right) = - \frac{\partial w}{\partial t} \geq 0, \quad \text{at } q = q^*(t, y). \tag{A.7}$$

Hence (A.6) is equivalent to (A.3). Q.E.D.

PROPOSITION 3. *Every local maximum (minimum) of the utility function $W(q, y, y)$ with respect to y , is a local maximum (minimum) of the function $q^*(y, y)$ and of the function $CS(y)$.*

PROOF. Since $q^*(y, y)$ maximizes $W(q, y, y) - R(q)$ with respect to q , it must satisfy the necessary conditions

$$w(q, y, y) - R'(q) = 0 \tag{A.8}$$

and

$$\partial w(q, y, y) / \partial q - R''(q) \leq 0, \tag{A.9}$$

for $q = q^*(y, y) > 0$.

Since (A.8) defines $q^*(y, y)$ implicitly, it follows that

$$\frac{dq^*(y, y)}{dy} = - \frac{\{\partial w(q, y, y) / \partial y\}}{\{\partial w(q, y, y) / \partial q - R''(q)\}} \Bigg|_{q=q^*(y, y)} \tag{A.10}$$

where

$$\frac{\partial w(q, y, y)}{\partial y} = \frac{\partial}{\partial q} \left\{ \frac{\partial W(q, y, y)}{\partial y} \right\} \tag{A.11}$$

and

$$\frac{\partial W(q, y, y)}{\partial y} = \left\{ \frac{\partial W(q, t, y)}{\partial t} + \frac{\partial W(q, t, y)}{\partial y} \right\} \Bigg|_{t=y} \tag{A.12}$$

Since the denominator in (A.10) is nonpositive by (A.9), $dq^*(y, y)/dy$ has the same sign as $[\partial w(q, y, y) / \partial y]_{q=q^*(y, y)}$.

However, since the local extrema of $W(q, y, y)$ are independent of q , the local extrema of $W(q, y, y)$ for any q coincide with those of $w(q, y, y)$, which are also local extrema of $q^*(y, y)$.

From (3.3) we have

$$\begin{aligned} \frac{dCS(y)}{dy} = & \{w(q^*(y, y), y, y) - R'(q^*(y, y))\} \frac{dq^*(y, y)}{dy} \\ & + \left. \frac{\partial W(q, y, y)}{\partial y} \right|_{q=q^*(y, y)}. \end{aligned} \quad (\text{A.13})$$

If $q^*(y, y) = 0$, then $CS(y) = W(0, y, y) = 0$, so the two functions are identical. Otherwise, if $q^*(y, y) > 0$, then the first term in (A.13) vanishes by (A.8), so the derivatives of $CS(y)$ and $W(q, y, y)$ with respect to y have the same signs and vanish at the same values of y . Q.E.D.

PROPOSITION 4. *The equation $CS(y) = 0$ will have a root $y_1 \in [0, 1]$ such that $CS'(y_1) > 0$ if and only if there exists an adjacent root $y_2 \in (y_1, 1]$ such that $CS'(y_2) < 0$. Furthermore, for every such pair of roots, $CS(y) \geq 0$ for $y \in [y_1, y_2]$.*

PROOF. The assumption that $W(q, 1, y) = W(q, t, 0) = 0$, implies

$$CS(0) = CS(1) = \text{Sup}_{q>0} \{-R(q)\} = -R(0^+) \leq 0.$$

Thus, since $CS(y)$ is continuous, and nonpositive at both ends of the interval $[0, 1]$, it can cross the zero axis in that interval only an even number of times and the roots can be paired as stated. Q.E.D.

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